

Local Ramsey theory. An abstract approach

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Abstract

It is shown that the known notion of selective coideal can be extended to a family \mathcal{H} of subsets of \mathcal{R} , where (\mathcal{R}, \leq, r) is a topological Ramsey space in the sense of Todorćević (see [15]). Then it is proven that, if \mathcal{H} selective, the \mathcal{H} -Ramsey and \mathcal{H} -Baire subsets of \mathcal{R} are equivalent. This extends the results of Farah in [5] for semiselective coideals of \mathbb{N} . Also, it is proven that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation.

1 Introduction

In [8], Mathias introduces the *happy families* (or selective coideals) of subsets of \mathbb{N} and relativizes the notion of completely Ramsey (see [6]) subsets of $\mathcal{P}(\mathbb{N})$ (the set of subsets of \mathbb{N}) to such families. Then he proves that analytic sets are \mathcal{F} -Ramsey when \mathcal{F} is a Ramsey ultrafilter and generalizes this result for arbitrary happy families. In [5], Farah gives an answer to the question of Todorćević: what are the combinatorial properties of the family \mathcal{H} of ground model subsets of \mathbb{N} which warrant diagonalization of the Borel partitions? This is done by imposing a condition on \mathcal{H} which is weaker than selectivity, that is the notion of *semiselectivity*. In that work he proves that the semiselectivity of \mathcal{H} is enough for a subset of $\mathbb{N}^{[\infty]}$ to be \mathcal{H} -Ramsey if and only if it has the (abstract) Baire property with respect to \mathcal{H} . In [10], Mijares extends this result to any *topological Ramsey space* (see [15]) by generalizing the notion of Ramsey ultrafilter to such spaces. In this work, it is proven that a family \mathcal{H} of subsets of a topological Ramsey space \mathcal{R} , provided with suitable features, corresponds to

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the semiselective coideal given by Farah in the above mentioned work. Then the results about \mathcal{H} -Ramsey and \mathcal{H} -Baire sets of \mathcal{R} are extended to this context.

The structure of this work is as follows: In section 2, some material regarding the so called *topological Ramsey theory* is given, see [15] or [2]. In section 3 it is proven that certain features of a family of sets of \mathcal{R} are sufficient for a subset of \mathcal{R} to be \mathcal{H} -Ramsey if and only if it is \mathcal{H} -Baire. This is done by defining the *D-O property* which is proposed as the corresponding notion to dense open sets. This is what is known as *local Ramsey theory*. In section 4, it is shown that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation if \mathcal{H} is selective. Finally, in section 5, examples for which the results hold are given.

2 Preliminaries: Topological Ramsey Theory.

The definitions and results throughout this section are expected to appear in [15]. A previous presentation of the following notions can also be found in [2].

Consider a triplet of the form (\mathcal{R}, \leq, r) , where \mathcal{R} is a set, \leq is a quasi order on \mathcal{R} , \mathbb{N} is the set of natural numbers, and $r : \mathbb{N} \times \mathcal{R} \rightarrow \mathcal{AR}$ is a function with range \mathcal{AR} . For each $A \in \mathcal{R}$, we say that $r_n(A)$ is the *n*th approximation of A . Denote, for every $n \in \mathbb{N}$ and every $A \in \mathcal{R}$, $r(n, A) = r_n(A)$ and $r_n(\mathcal{R}) = \mathcal{AR}_n$. In order to capture the combinatorial structure required to ensure the provability of an Ellentuck type theorem, some assumptions on (\mathcal{R}, \leq, r) will be imposed. The first three of them are the following:

- (A.1) For any $A \in \mathcal{R}$, $r_0(A) = \emptyset$.
- (A.2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n) r_n(A) \neq r_n(B)$.
- (A.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n) r_i(A) = r_i(B)$.

These three assumptions allow us to identify each $A \in \mathcal{R}$ with the sequence $(r_n(A))_n$ of its approximations. In this way, if \mathcal{AR} has the discrete topology, \mathcal{R} can be identified with a subspace of the (metric) space $\mathcal{AR}^{[\infty]}$ (with the product topology) of all the sequences of elements of \mathcal{AR} . Via this identification, \mathcal{R} will be regarded as a subspace of $\mathcal{AR}^{[\infty]}$, and we will say that \mathcal{R} is *metrically closed* if it is a closed subspace of $\mathcal{AR}^{[\infty]}$.

Also, for $a \in \mathcal{AR}$, define the *length* of a , $|a|$, as the unique n such that $a = r_n(A)$ for some $A \in \mathcal{R}$, and the *Ellentuck type neighborhoods* on \mathcal{R}

$$[a, A] = \{B \in \mathcal{R} : (\exists n)(a = r_n(B)) \text{ and } B \leq A\}$$

where $a \in \mathcal{AR}$ and $A \in \mathcal{R}$. If $[a, A] \neq \emptyset$ we will say that a is *compatible* with A (or A is compatible with a). Let $\mathcal{AR}[A] = \{a \in \mathcal{AR} : a \text{ is compatible with } A\}$.

Denote $[n, A]$ for $[r_n(A), A]$, and $\text{Exp}(\mathcal{R})$ for the family of all the neighborhoods $[n, A]$. This family generates the natural "exponential" topology on \mathcal{R} which is finer than the product topology.

Now, an analog notion for subsets of \mathcal{R} , to that of *Ramseyness* for subsets of $\mathbb{N}^{[\infty]}$ is defined:

Definition 1. A set $\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey** if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey null** if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Definition 2. We say that (\mathcal{R}, \leq, r) is a **Ramsey space** if subsets of \mathcal{R} with the Baire property are Ramsey and meager subsets of \mathcal{R} are Ramsey null.

In [15] it is shown that (A.1), (A.2) and (A.3), together with the following three assumptions are conditions of sufficiency for a triplet (\mathcal{R}, \leq, r) , with \mathcal{R} metrically closed, to be a Ramsey space:

(A.4) (*Finitization*) There is a quasi order \leq_{fin} on \mathcal{AR} such that:

- (i) $A \leq B$ iff $\forall n \exists m \ r_n(A) \leq_{fin} r_m(B)$.
- (ii) $\{b \in \mathcal{AR} : b \leq_{fin} a\}$ is finite, for every $a \in \mathcal{AR}$.

Given a and A , we define the *depth of a in A* , $depth_A(a)$, as the minimal n such that $a \leq_{fin} r_n(A)$.

(A.5) (*Amalgamation*) Given compatible a and A with $depth_A(a) = n$, the following holds:

- (i) $\forall B \in [n, A] \ ([a, B] \neq \emptyset)$.
- (ii) $\forall B \in [a, A] \ \exists A' \in [n, A] \ ([a, A'] \subseteq [a, B])$.

(A.6) (*Pigeon Hole Principle*) Given compatible a and A with $depth_A(a) = n$, for each partition $\phi : \mathcal{AR}_{|a|+1} \rightarrow \{0, 1\}$ there is $B \in [n, A]$ such that ϕ is constant in $r_{|a|+1}[a, B]$.

Abstract Ellentuck Theorem:

Theorem 1 (Carlson). *Any (\mathcal{R}, \leq, r) with \mathcal{R} metrically closed and satisfying (A.1)-(A.6) is a Ramsey space.*

□

For instance, take $\mathcal{R} = \mathbb{N}^{[\infty]}$, the set of infinite subsets of \mathbb{N} , $\leq = \subseteq$ and $r_n(A)$ = the first n elements of A , for each $A \in \mathbb{N}^{[\infty]}$. So, the set of approximations is $\mathcal{AR} = \mathbb{N}^{[<\infty]}$, the set of finite subsets of \mathbb{N} . The family of neighborhoods $[a, A]$, with $a \in \mathbb{N}^{[<\infty]}$ and $A \in \mathbb{N}^{[\infty]}$, is the family of Ellentuck neighborhoods. Define \leq_{fin} as $a \leq_{fin} b$ iff $(a = b = \emptyset \text{ or } a \subseteq b \text{ and } \max(a) = \max(b))$, for $a, b \in \mathbb{N}^{[<\infty]}$. With these definitions, (A.1)-(A.6) hold. In this case (A.6) reduces to a natural variation of the classical pigeon hole principle for finite partitions of an infinite

set of natural numbers. Note also that $\mathbb{N}^{[\infty]}$ is easily identified with a closed subspace of $\mathcal{AR}^{[\infty]}$, namely, the set of all the sequences $(x_n)_n$ of finite sets such that $x_n = x_{n+1} \setminus \{\max(x_{n+1})\}$, for each $n \in \mathbb{N}$. Then $(\mathbb{N}^{[\infty]}, \subseteq, r)$ is a Ramsey space in virtue of the abstract Ellentuck theorem. Hence, Ellentuck's theorem is obtained as a corollary:

Corollary 1 (Ellentuck). *Given $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, the following hold:*

- (a) \mathcal{X} is Ramsey iff \mathcal{X} has the Baire Property, relative to Ellentuck's topology.
- (b) \mathcal{X} is Ramsey null iff \mathcal{X} is meager, relative to Ellentuck's topology.

□

3 Selectivity

From now on suppose that (\mathcal{R}, \leq, r) is a topological Ramsey space; that is, (A.1) ... (A.6) hold and \mathcal{R} is metrically closed. The following features are inspired on the known notion of coideal, so it will be used the same name: we say that $\mathcal{H} \subseteq \mathcal{R}$ is a **coideal** of \mathcal{R} (or simply a *coideal*) if it satisfies:

1. If $A \leq B$ and $A \in \mathcal{H}$ then $B \in \mathcal{H}$.
2. **(A.5) mod \mathcal{H} :** Given $A \in \mathcal{H}$ and $a \in \mathcal{AR}(A)$, if $\text{depth}_A(a) = n$, then:
 - i) $\forall B \in [n, A] \cap \mathcal{H} ([a, B] \cap \mathcal{H} \neq \emptyset)$.
 - ii) $\forall B \in [a, A] \cap \mathcal{H} \exists A' \in [n, A] \cap \mathcal{H} ([a, A'] \subseteq [a, B])$.
3. **(A.6) mod \mathcal{H} :** Given $a \in \mathcal{AR}$ with length l and $\mathcal{O} \subseteq \mathcal{AR}_{l+1}$. Then, for every $A \in \mathcal{H}$ with $[a, A] \neq \emptyset$, there exists $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{l+1}([a, B]) \subseteq \mathcal{O}$ or $r_{l+1}([a, X]) \subseteq \mathcal{O}^c$.

The natural definitions of \mathcal{H} -Ramsey and \mathcal{H} -Baire sets will be:

Definition 3. $\mathcal{X} \subseteq \mathcal{R}$ is **\mathcal{H} -Ramsey** if for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $B \in [a, A] \cap \mathcal{H}$ with $[a, B] \neq \emptyset$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$. If for every $[a, A] \neq \emptyset$, there exists $B \in [a, A] \cap \mathcal{H}$ with $[a, B] \neq \emptyset$ such that $[a, B] \subseteq \mathcal{X}^c$; we say that \mathcal{X} is **\mathcal{H} -Ramsey null**.

Definition 4. $\mathcal{X} \subseteq \mathcal{R}$ is **\mathcal{H} -Baire** if for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b, B] \subseteq [a, A]$, with $B \in \mathcal{H}$, such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \subseteq \mathcal{X}^c$. If for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b, B] \subseteq [a, A]$, with $B \in \mathcal{H}$, such that $[b, B] \subseteq \mathcal{X}^c$; we say that \mathcal{X} is **\mathcal{H} -meager**.

It is clear that if $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{H} -Ramsey then \mathcal{X} is \mathcal{H} -Baire. Now, the notion corresponding to *dense open* sets will be defined in this context: Given $A \in \mathcal{H}$ and $\mathcal{I} \subseteq \mathcal{AR}(A)$, we say that the sequence $(\mathcal{D}_a)_{a \in \mathcal{I}}$, with $\mathcal{D}_a \subseteq \mathcal{H}$, $[a, C] \neq \emptyset$ for some $C \in \mathcal{D}_a$ and every $a \in \mathcal{I}$, has the **D-O property bellow A** if for every $a \in \mathcal{I}$ the following hold:

1. $\forall B \in [a, A] \cap \mathcal{H} \exists C \in \mathcal{D}_a (C \leq B)$.
2. $[B \in \mathcal{D}_a \upharpoonright A \wedge (C \in [\text{depth}_B(a), B]) \cap \mathcal{H}] \Rightarrow C \in \mathcal{D}_a$.

The notion of selectivity is clear in this context:

Definition 5. A coideal $\mathcal{H} \subseteq \mathcal{R}$ is **selective** if given $A \in \mathcal{H}$ and $(A_a)_{a \in \mathcal{I}}$, with $\mathcal{I} \subseteq \mathcal{AR}$, $A_a \in \mathcal{H} \upharpoonright A$ and $[a, A_a] \neq \emptyset$ for $a \in \mathcal{I}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that $[a, B] \subseteq [a, A_a]$ for every $a \in \mathcal{I} \cap \mathcal{AR}(B)$.

Now, it will be shown that selectivity implies the following property which will be useful in proving the main result of this work. The same name of the corresponding notion of coideals on \mathbb{N} introduced by Farah will be used.

Definition 6. We say that $\mathcal{H} \subseteq \mathcal{R}$ is **semiselective** if given $A \in \mathcal{H}$, for every sequence $(\mathcal{D}_a)_{a \in \mathcal{I}}$ with $\mathcal{I} \subseteq \mathcal{AR}(A)$, $\mathcal{D}_a \subseteq \mathcal{H}$ and with the D-O property below A , there exists $B \in \mathcal{H} \upharpoonright A$ such that $[\text{depth}_B(a), B] \cap \mathcal{H} \subseteq \mathcal{D}_a$ for every $a \in \mathcal{I} \cap \mathcal{AR}(B)$.

Proposition 1. If $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal then \mathcal{H} is semiselective.

Proof: Given $A \in \mathcal{H}$, consider $(\mathcal{D}_a)_{a \in \mathcal{I}}$ with $\mathcal{I} \subseteq \mathcal{AR}(A)$, $\mathcal{D}_a \subseteq \mathcal{H}$ and with the D-O property below A . For $a \in \mathcal{AR}(A)$, by (A.5) mod \mathcal{H} there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \cap \mathcal{H} \neq \emptyset$. By the D-O property, we can choose $A_a \in \mathcal{D}_a$ with $A_a \leq A$ and (again, by (A.5) mod \mathcal{H}) $[a, A_a] \neq \emptyset$. By selectivity, there exists $B \in \mathcal{H} \upharpoonright A$ such that $[a, B] \subseteq [a, A_a]$ for $a \in \mathcal{AR}(B) \cap \mathcal{I}$. But $[A_a \in \mathcal{D}_a \upharpoonright A \wedge (C \in [\text{depth}_B(a), B]) \cap \mathcal{H}] \Rightarrow C \in \mathcal{D}_a$ (D-O property). Thus, $[\text{depth}_B(a), B] \cap \mathcal{H} \subseteq \mathcal{D}_a$ for every $a \in \mathcal{AR}(B) \cap \mathcal{I}$. \square

The following is the version of theorem 1.6 from [10] corresponding to this context and can be easily generalized to partitions in n pieces:

Theorem 2. Suppose that $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal. Then, given a partition $f: \mathcal{AR}_2 \rightarrow \{0, 1\}$ and $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that f is constant on $\mathcal{AR}_2(B)$.

Proof: Let f be the partition $\mathcal{AR}_2 = \mathcal{C}_0 \cup \mathcal{C}_1$, and consider $A \in \mathcal{H}$. By (A.6) mod \mathcal{H} , for every $a \in \mathcal{AR}(A)$ we can define the nonempty

$$\mathcal{D}_a = \{B \in [\text{depth}_A(a), A] \cap \mathcal{H} : f \text{ is constant on } r_2([a, B])\}$$

if $a \in \mathcal{AR}_1(A)$, and $\mathcal{D}_a = \mathcal{H} \upharpoonright A$ otherwise; which gives us a sequence with the D-O property below A . By selectivity (or the S-property), we have $\hat{B} \in \mathcal{H} \upharpoonright A$ such that $[\text{depth}_{\hat{B}}(a), \hat{B}] \subseteq \mathcal{D}_a$ for every $a \in \mathcal{AR}_1(\hat{B})$. Since $\hat{B} \in [\text{depth}_{\hat{B}}(a), \hat{B}] \subseteq \mathcal{D}_a$ for every $a \in \mathcal{AR}_1(\hat{B})$, there exists $i_a \in \{0, 1\}$ such that $b \in \mathcal{C}_{i_a}$ if $b \in r_2([a, \hat{B}])$. Now, consider the partition $g: \mathcal{AR}_1 \rightarrow \{0, 1\}$ defined by $g(a) = i_a$ if $a \in \mathcal{AR}_1(\hat{B})$. By (A.6) mod \mathcal{H} , there exists $B \in [0, \hat{B}] \cap \mathcal{H}$ such that g is constant on $r_1([0, B]) = \mathcal{AR}_1(B)$. But $B \leq \hat{B} \leq A$, so B is as required. \square

To give the local version of the corresponding Galvin lemma (or Nash-williams theorem) for selective coideals of \mathcal{R} , the following combinatorial forcing will be used: Fix $\mathcal{F} \subseteq \mathcal{AR}$. We say that $A \in \mathcal{H}$ **accepts** $a \in \mathcal{AR}$ if $[a, A] = \emptyset$ or for every $B \in [a, A] \cap \mathcal{H}$ there exists $n \in \mathbb{N}$ such that $r_n(B) \in \mathcal{F}$. We say that A **rejects** a if $[a, A] \neq \emptyset$ and no element of $[depth_A(a), A] \cap \mathcal{H}$ accepts a ; and we say that A **decides** a if A either accepts or rejects a . This combinatorial forcing has the following properties:

1. If A accepts a , then every $B \in \mathcal{H} \restriction A$ accepts a .
2. If A rejects a , then every $B \in \mathcal{H} \restriction A$ rejects a , if $[a, B] \neq \emptyset$.
3. For every $A \in \mathcal{H}$ and every $a \in \mathcal{AR}(A)$ there exists $B \in [depth_A(a), A] \cap \mathcal{H}$ which decides a .
4. If A accepts a then A accepts every $b \in r_{|a|+1}([a, A])$.
5. If A rejects a then there exists $B \in [depth_A(a), A] \cap \mathcal{H}$ which rejects every $b \in r_{|a|+1}([a, B])$.

Claim 1: Given $A \in \mathcal{H}$, with \mathcal{H} selective, there exists $B \in \mathcal{H} \restriction A$ which decides every $b \in \mathcal{AR}(B)$.

Proof: For every $a \in \mathcal{AR}(A)$ define

$$\mathcal{D}_a = \{C \in [depth_A(a), A] \cap \mathcal{H} : C \text{ decides } a\}$$

Then $(\mathcal{D}_a)_{a \in \mathcal{AR}(A)}$ has the D-O property, so there exists $B \in \mathcal{H} \restriction A$ such that for every $a \in \mathcal{AR}(B)$ we have $[depth_B(a), B] \cap \mathcal{H} \subseteq \mathcal{D}_a$. Thus, B decides every $a \in \mathcal{AR}(B)$. □

Lemma 1. Given $\mathcal{F} \subseteq \mathcal{AR}$, a selective coideal $\mathcal{H} \subseteq \mathcal{R}$, and $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \restriction A$ such that one of the following holds:

1. $\mathcal{AR}(B) \cap \mathcal{F} = \emptyset$, or
2. $\forall C \in \mathcal{H} \restriction B (\exists n \in \mathbb{N}) (r_n(C) \in \mathcal{F})$.

Proof: consider B as in claim 1. If B accepts \emptyset part (2) holds. Assume that B rejects \emptyset and for $a \in \mathcal{AR}(B)$ define

$$\mathcal{D}_a = \{C \in [depth_B(a), B] \cap \mathcal{H} : C \text{ rejects every } b \in r_{|a|+1}([a, C])\}$$

if B rejects a , and $\mathcal{D}_a = \mathcal{H} \restriction B$ otherwise. So, $(\mathcal{D}_a)_{a \in \mathcal{AR}(A)}$ has the D-O property below A . Then we have $\hat{B} \in \mathcal{H} \restriction B$ such that $[depth_{\hat{B}}(a), \hat{B}] \cap \mathcal{H} \subseteq \mathcal{D}_a$ for every $a \in \mathcal{AR}(\hat{B})$. By induction on the length, \hat{B} rejects every $a \in \mathcal{AR}(\hat{B})$, hence no element of $\mathcal{AR}(\hat{B})$ is in \mathcal{F} . Thus, part (1) holds. □

Theorem 3. If $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal then $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{H} -Ramsey iff \mathcal{X} is \mathcal{H} -Baire

Proof: Let \mathcal{X} be a \mathcal{H} -Baire subset of \mathcal{R} and consider $A \in \mathcal{H}$. As before, we only proof the result for $[\emptyset, A]$ without loss of generality. For $a \in \mathcal{AR}(A)$ define

$$\mathcal{D}_a = \{B \in [\text{depth}_A(a), A] \cap \mathcal{H} : [a, B] \subseteq \mathcal{X} \text{ or } [a, B] \subseteq \mathcal{X}^c \\ \text{or } ([a, C] \not\subseteq \mathcal{X} \text{ and } [a, C] \not\subseteq \mathcal{X}^c \forall C \in [a, B])\}$$

Then $(\mathcal{D}_a)_a$ has the D-O property below A . Let $\hat{B} \in \mathcal{H} \upharpoonright A$ such that, for $a \in \mathcal{AR}(\hat{B})$, $[\text{depth}_{\hat{B}}(a), \hat{B}] \cap \mathcal{H} \subseteq \mathcal{D}_a$. Let $\mathcal{F}_0 = \{a \in \mathcal{AR}(A) : [a, \hat{B}] \subseteq \mathcal{X}\}$ and $\mathcal{F}_1 = \{a \in \mathcal{AR}(A) : [a, \hat{B}] \subseteq \mathcal{X}^c\}$. By applying lemma 1 to \mathcal{F}_0 (or to \mathcal{F}_1) and \hat{B} , we obtain $B \in \mathcal{H} \upharpoonright \hat{B}$ such that $[\emptyset, B] \subseteq \mathcal{X}$ (or $[\emptyset, B] \subseteq \mathcal{X}^c$) or $\mathcal{AR}(B) \cap (\mathcal{F}_0 \cup \mathcal{F}_1) = \emptyset$. The latter case is not possible: since \mathcal{X} is \mathcal{H} -Baire, there exists $\emptyset \neq [b, C] \subseteq [\emptyset, B]$ such that $[b, C] \subseteq \mathcal{X}$ or $[b, C] \subseteq \mathcal{X}^c$. By (A.5) mod \mathcal{H} , we can suppose that $C \in [b, C] \subseteq [b, \hat{B}]$, and since $\hat{B} \in \mathcal{D}_b$, we conclude that $b \in \mathcal{F}_0 \cup \mathcal{F}_1$. The reverse implication is obvious. \square

Now, we give one more application of lemma 1. Recall that the *metric open* subsets of \mathcal{R} are of the form

$$[b] = \{A \in \mathcal{R} : b \sqsubseteq A\}$$

where $b \sqsubseteq A$ means $\exists n \in \mathbb{N}(r_n(A) = b)$.

Theorem 4. *Suppose that $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal. Then the metric open subsets of \mathcal{R} are \mathcal{H} -Ramsey.*

Proof: Let $\mathcal{X} \subseteq \mathcal{R}$ metric open and consider $[\emptyset, A]$ with $A \in \mathcal{H}$. Define, for every $a \in \mathcal{AR}$

$$\mathcal{D}_a = \{B : [a, B] \subseteq \mathcal{X} \text{ or } [a, B] \subseteq \mathcal{X}^c \text{ or } \forall C \leq B ([a, C] \not\subseteq \mathcal{X} \text{ and } [a, C] \not\subseteq \mathcal{X}^c)\}$$

Thus, $\{\mathcal{D}_a\}_a$ has the D-O property below A . So, let $B \in \mathcal{H} \upharpoonright A$ be such that $[\text{depth}_A(a), B] \cap \mathcal{H} \subseteq \mathcal{D}_a$ for every $a \in \mathcal{AR}(B)$. Now, define $h : \mathcal{AR} \rightarrow \{0, 1, 2\}$ by

$$h(a) = \begin{cases} 0 & \text{if } [a, B] \subseteq \mathcal{X} \\ 1 & \text{if } [a, B] \subseteq \mathcal{X}^c \\ 2 & \text{otherwise} \end{cases}$$

If $h(a) = 2$, by restricting h to $\mathcal{AR}_{|a|+1}$ we obtain (by (A.6) mod \mathcal{H}) $C_a \in [\text{depth}_B(a), B] \cap \mathcal{H}$ such that h is constant on $r_{|a|+1}([a, C_a])$. Furthermore, that constant is 2 since $h(a) = 2$ and $B \in \mathcal{D}_a$.

Claim 1. *If $h(\emptyset) = 2$ then there exists $C \in \mathcal{H} \upharpoonright B$ such that $\forall b \in \mathcal{AR}(C)(h(b) = 2)$.*

Proof:(of the claim) Define for $b \in \mathcal{AR}$, C_b as before if $[b, B] \neq \emptyset$ and $C_b = B$ otherwise. Then $\{C_b\}_b$ has the D-O property. Let $C \in \mathcal{H} \upharpoonright B$ be such that $[\text{depth}_C(b), C] \cap \mathcal{H} \subseteq \mathcal{C}_a$ for every $b \in \mathcal{AR}(C)$. Suppose that $h(b) \neq 2$ for some $b = r_{|b|}(B) \in \mathcal{AR}(B)$ and choose it with minimal depth in C . Thus, $b \neq \emptyset$

since $h(\emptyset) = 2$. Let $b' = r_{|b|-1}(\hat{B})$. Then $h(b') = 2$, but $b \in r_{|b'|+1}([b', C]) \subseteq r_{|b'|+1}([b', C_{b'}])$ and hence $h(b) = 2$ (see the paragraph before the claim). This is a contradiction, and the claim is proved. \square

Now it will be shown that $h(\emptyset) < 2$. Suppose that $h(\emptyset) = 2$ and C is as in the claim. Then $[\emptyset, C] \not\subseteq \mathcal{X}$ and $[\emptyset, C] \not\subseteq \mathcal{X}^c$. Consider $\hat{C} \in \mathcal{X} \upharpoonright C$. Since \mathcal{X} is metric open, there exists $b \in \mathcal{AR}$ such that $b \sqsubseteq B$ and $[b] \subseteq \mathcal{X}$, i. e., $h(b) = 0$, which is a contradiction (by the claim). This completes the proof of theorem. \square

4 The Souslin operation

The goal of this section is to show that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation when \mathcal{H} is a selective coideal.

Lemma 2. *If $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal of \mathcal{R} then the families of \mathcal{H} -Ramsey and \mathcal{H} -Ramsey null subsets of \mathcal{R} are closed under countable union.*

Proof: Fix $A \in \mathcal{H}$. The proof will be made for $[\emptyset, A]$ without loss of generality. Suppose that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{H} -Ramsey null subsets of \mathcal{R} . Define for $a \in \mathcal{AR}(A)$

$$\mathcal{D}_a = \{B \in [a, A] \cap \mathcal{H} : [a, B] \subseteq \mathcal{X}_n^c \ \forall n \leq |a|\}$$

Then $(\mathcal{D}_a)_a$ has the D-O property below A , so let $B \in \mathcal{H} \upharpoonright A$ be such that $[\text{depth}_B(a), B] \cap \mathcal{H} \subseteq \mathcal{D}_a$ for all $a \in \mathcal{AR}(B)$. Thus, $[\emptyset, B] \subseteq \bigcap \mathcal{X}_n^c$ (since $B \in [\text{depth}_B(a), B] \cap \mathcal{H}$ for every $a \in \mathcal{AR}(B)$). Now, suppose that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{H} -Ramsey subsets of \mathcal{R} and consider $\emptyset \neq [a, A]$. If there exists $B \in \mathcal{H} \upharpoonright A$ such that $[a, B] \subseteq \mathcal{X}_n$ for some n , we are done. Otherwise, using an argument similar to the one above, we prove that $\bigcup \mathcal{X}_n$ is \mathcal{H} -Ramsey null. \square

Recall that given a set X , two subsets A, B of X are "compatibles" with respect to a family \mathcal{F} of subsets X if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. And \mathcal{F} is *M-like* if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of \mathcal{F} which is not compatible with any member of \mathcal{G} is compatible with $X \setminus \bigcup \mathcal{G}$. A σ -algebra \mathcal{A} of subsets of X together with a σ -ideal $\mathcal{A}_0 \subseteq \mathcal{A}$ is a *Marczewski pair* if for every $A \subseteq X$ there exists $\Phi(A) \in \mathcal{A}$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in \mathcal{A} \rightarrow B \in \mathcal{A}_0$. The following is a well known fact:

Theorem 5 (Marczewski). *Every σ -algebra of sets which together with a σ -ideal is a Marczewski pair, is closed under the Souslin operation.* \square

Denote $\text{Exp}(\mathcal{H}) = \{[n, A] : n \in \mathbb{N}, A \in \mathcal{H}\}$. \mathcal{H} selective coideal of \mathcal{R} .

Proposition 2. If $|\mathcal{H}| = 2^{\aleph_0}$, then the family $\text{Exp}(\mathcal{H})$ is *M-like*.

Proof: Consider $\mathcal{B} \subseteq \text{Exp}(\mathcal{H})$ with $|\mathcal{B}| < |\text{Exp}(\mathcal{H})| = 2^{\aleph_0}$ and suppose that $[a, A]$ is not compatible with any member of \mathcal{B} , i. e. for every $B \in \mathcal{B}$, $B \cap [a, A]$ does not contain any member of $\text{Exp}(\mathcal{H})$. We claim that $[a, A]$ is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$. In fact:
Since $|\mathcal{B}| < 2^{\aleph_0}$, $\bigcup \mathcal{B}$ is \mathcal{H} -Baire (it is \mathcal{H} -Ramsey). So, there exist $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that:

1. $[b, B] \subseteq \bigcup \mathcal{B}$ or
2. $[b, B] \subseteq \mathcal{R} \setminus \bigcup \mathcal{B}$

(1) is not possible because $[a, A]$ is not compatible with any member of \mathcal{B} . And (2) says that $[a, A]$ is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$ □

Proposition 2 says that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} together with the family of \mathcal{H} -Ramsey null subsets of \mathcal{R} is a Marczewski pair (see section 2 of [14]). Thus, by theorem 5, we have the result:

Theorem 6. *The family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation.* □

As a consequence of theorems 4 and 6, the following holds:

Theorem 7. *Suppose that $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal. Then the analytic subsets of \mathcal{R} are \mathcal{H} -Ramsey.* □

5 Examples

The goal of this section is to give examples of topological Ramsey spaces for which the previous results are illustrated. An example from the Ellentuck's topological Ramsey space $(\mathbb{N}^{[\infty]}, \subseteq, r_n)$ as defined in section 2 is the following: fix $(x_n)_n \subseteq \mathbb{N}^{\mathbb{N}}$ and x a cluster point of $(x_n)_n$. Define

$$\mathcal{H} = \{A \in \mathbb{N}^{[\infty]} : x \text{ is cluster point of } (x_n)_{n \in A}\}$$

Then, \mathcal{H} is a coideal (in our context and in the sense of the known notion of coideal). If x_n is borel for every $n \in \mathbb{N}$, then \mathcal{H} is selective and $\mathbb{N}^{[\infty]} \setminus \mathcal{H}$ is analytic, that is to say, \mathcal{H} is Π_1^1 .

Another example: Fix $k \in \mathbb{N}$. Given $p: \mathbb{N} \rightarrow \{0, 1, \dots, k\}$, denote $\text{supp}(p) = \{n: p(n) \neq 0\}$ and $\text{rank}(p)$ the image set of p . Consider the set

$$\text{FIN}_k = \{p: \mathbb{N} \rightarrow \{0, 1, \dots\} : |\text{supp}(p)| < \infty \text{ and } k \in \text{rank}(p)\}$$

we say that $X = (x_n)_{n \in \mathcal{I}} \subseteq \text{FIN}_k$, with $\mathcal{I} \in \mathcal{P}(\mathbb{N})$ is a *basic block sequence* if

$$n < m \Rightarrow \max(\text{supp}(x_n)) < \min(\text{supp}(x_m))$$

For infinite sequences we assume that $\mathcal{I} = \mathbb{N}$. Define $T: FIN_k \rightarrow FIN_{k-1}$ by

$$T(p)(n) = \max\{p(n) - 1, 0\}$$

For $j \in \mathbb{N}$, $T^{(j)}$ is the j -th iteration of T . Given a basic block sequence $X = (x_n)_{n \in \mathcal{I}}$ we define $[X] \subseteq FIN_k$ as the set which elements are of the form

$$T^{(j_0)}(x_{n_0}) + T^{(j_1)}(x_{n_1}) + \dots + T^{(j_r)}(x_{n_r})$$

with $n_0 < n_1 < \dots < n_r \in \mathcal{I}$, $j_0 < j_1 < \dots < j_r \in \{0, 1, \dots, k\}$, and $j_i = 0$ for some $i \in \{0, 1, \dots, r\}$. Denote $FIN_k^{[\infty]}$, the set of infinite basic block sequences, for $A, B \in FIN_k^{[\infty]}$, define

$$A \leq B \Leftrightarrow A \subseteq [B]$$

and $r_n(A)$ = "the first n elements of A ". Then $(FIN_k^{[\infty]}, \leq, r)$ is a topological Ramsey space (see [15]). Furthermore, we have the following well known result:

Theorem 8 (Gowers). *Given an integer $n > 0$ and $f: FIN_k \rightarrow \{0, 1, \dots, r-1\}$, there exists $A \in FIN_k^{[\infty]}$ such that f is constant on $[A]$.*

□

For $k = 1$, the previous theorem reduces to the famous Hindman's theorem ([7]). Assuming **CH**, we define a well order $(\mathcal{P}(FIN_k), <)$, and for a fixed $X \subseteq FIN_k$ we find $A_X \in FIN^{[\infty]}$ such that

1. $\mathcal{AR}(A_X) \subseteq X$ or $\mathcal{AR}(A_X) \subseteq X^c$.
2. $X < Y \Rightarrow A_X \leq^* A_Y$

Where " $A \leq^* B$ " means that $A \leq B$ "from some n on". Suppose that we have defined A_Y for every $Y < X$. We only have to consider the case in which X is limit. If we have already $A_{Y_0} \geq A_{Y_1} \geq \dots$ for the predecessors Y_0, Y_1, \dots of X , we can choose $a_0 \in A_{Y_0}$, $a_1 \in A_{Y_1}$, ... such that $a_0 < a_1 < \dots$. Then, $A = a_0 \hat{\ } a_1 \hat{\ } \dots$ satisfies (2). Now, if

$$\mathcal{AR}(A) = (X \cap \mathcal{AR}(A)) \cup ((X^c \cap \mathcal{AR}(A)))$$

we can find, by theorem 8, $A_X \leq A$ which satisfies (1). It is clear that A_X satisfies (2) too. This completes the construction.

Now, the coideal:

$$\mathcal{H} = \{B \in FIN_k^{[\infty]} : \exists X \subseteq FIN_k (A_X \leq B)\}$$

It is clear that \mathcal{H} satisfies (1) and (2) from the definition of coideal. Now, by theorem 8, and the previous construction, given $B \in \mathcal{H}$ and $f: [B] \rightarrow \{0, 1\}$ there exists $B' \in \mathcal{H}$ such that f is constant on $[B']$. This is, (A.6)mod \mathcal{H} holds. The selectivity of \mathcal{H} is also a consequence of the construction of the A_X 's (which is strongly based on **CH**, of course. See [1]). The previous construction can be done on any topological Ramsey space in a similar way, under the assumption of **CH** or the Martin's axiom. That is to say, What is given above is a scheme of examples.

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